For most systems prohibitively high operating power levels are required to modify the gain characteristic by phonon saturation. In those exceptional systems where phonon Q's might be high enough to permit saturation at reasonable power levels, increases in phonon populations tend to transform the phonon-terminated maser into a ground-state maser. Depending upon whether the particular phonons are such that their vibrational structure borrows its intensity from distant electronic states or from the no-phonon line and provided the electronic populations are actually inverted, increases in phonon population can increase or decrease the amplitude of the gain characteristic at the operating frequency. Systems of both types will be very interesting to study, if it should prove possible to locate materials with saturable phonons.

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Ground-State Occupancy of an Ideal Bose-Einstein Gas Confined to a Finite Volume*

D. L. MILLS†
Department of Physics, University of California, Berkeley, California
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The number of particles in the ground state has been computed as a function of temperature for an ideal Bose-Einstein gas confined to a box of finite volume by evaluating the discrete sum over states on a computer. Large deviations from London's bulk-gas result are found when the length of the box is much greater than its width for the range of dimensions investigated here. It is shown that the deviations occur because in this limit the system tends to behave like a one-dimensional system.

I. INTRODUCTION

IT has been suggested[1] that the superfluid properties of liquid He^4 may be qualitatively understood by treating the fluid as a gas of noninteracting bosons. In the ideal-gas approximation the superfluid component is assumed to consist of the particles in the ground state, which is macroscopically occupied for temperatures below a critical temperature T_c.

Measurements of the temperature of the lambda point T_λ of liquid He^4 confined to fine pores have been performed.[2] A severe depression of T_λ was observed when the fluid was confined to pores of 40–50 Å diameter. In one case T_λ was depressed from the bulk value of 2.18–1.36°K in a sample consisting of pores of diameter 43 Å.

We have calculated the number of particles in the ground state as a function of temperature for an ideal Bose-Einstein gas confined to a container of width D and length L, where L ranges from 200–5000 Å and D is in the range of 10–100 Å in order to see if the ideal Bose-Einstein gas model is able to offer a qualitative understanding of the above mentioned depression of T_λ.

II. GENERAL DISCUSSION

Consider a system of n noninteracting bosons of spin zero and mass m. We assume the particles are confined to a box of length L and square cross section, where D is the length of a side of the square. The allowed single-particle energy levels are

$$E_{n_1n_2n_3}=\frac{\hbar^2}{2m}\left[\left(\frac{n_1\pi}{D}\right)^2+\left(\frac{n_2\pi}{D}\right)^2+\left(\frac{n_3\pi}{L}\right)^2\right],$$

where n_1, n_2 and n_3 range over the positive integers. The mean number of particles in the state (n_1,n_2,n_3) is

$$N_{n_1n_2n_3}=\left[\exp\left(\frac{E_{n_1n_2n_3}-\mu}{kT}\right)-1\right]^{-1},$$

where μ is the chemical potential, k is Boltzmann's constant and T is the temperature in degrees Kelvin. The chemical potential μ is determined from

$$N=\sum_{n_1,n_2,n_3=1}^{\infty}N_{n_1n_2n_3}=\sum_{n_1,n_2,n_3=1}^{\infty}\times\left[\exp\left(\frac{E_{n_1n_2n_3}-\mu}{kT}\right)-1\right]^{-1}. \quad (1)$$

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† National Science Foundation Predoctoral Fellow.
[1] F. London, Phys. Rev. **54**, 947 (1938).
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The number of particles in the ground state is

$$N_{111}=\left[\exp\left(\frac{E_{111}-\mu}{kT}\right)-1\right]^{-1}\cong\frac{kT}{E_{111}-\mu},$$

if $N_{111}\gg 1$. We write $N_{111}=\alpha(T)N$. Then Eq. (1) becomes

$$N=\sum_{n_1,n_2,n_3=1}^{\infty}\left[\exp\left(\frac{E_{n_1n_2n_3}-E_{111}}{kT}+\frac{1}{\alpha(T)N}\right)-1\right]^{-1}. \quad (2)$$

For given values of D, L, and N the fraction of the total number of particles in the ground state at temperature T may be found by solving Eq. (2) for $\alpha(T)$. If the average particle-number density is n, then $N=nD^2L$. In the limit as D and L approach infinity with n held fixed we have the well-known result[1]

$$\begin{aligned}\alpha(T)&=1-(T/T_c)^{3/2}, \quad T\leq T_c\\ &=\quad 0 \quad\quad, \quad T>T_c,\end{aligned} \quad (3)$$

where

$$T_c=2\pi\left(\frac{n}{2.612}\right)^{2/3}\frac{\hbar^2}{mk}.$$

If we employ the values $m=6.68\times10^{-24}$ g and $n=2.25\times10^{22}$ particles/cm^3 which are characteristic of liquid He^4, we have $T_c=3.165$ K°.

We have solved Eq. (2) for $\alpha(T)$ by carrying out the discrete sum over states for the values of D and L mentioned above. An IBM-7090 computer was utilized to evaluate the sums. The values of m and n given above were used. The results of the computations are shown in Figs. 1–3. The curves labeled "bulk gas" are plots of Eq. (3).

III. DISCUSSION OF RESULTS

We first discuss the results for $L=1000$ Å as presented in Fig. 2. It may be seen that for $D=100$ Å the system behaves very much like the bulk gas. However, when $D=10$ Å very different results are obtained. We see that for the latter case the ground state initially is depopulated very rapidly as the temperature is raised from absolute zero while at higher temperatures a slowly-varying nonexponential tail remains. It will be shown below that these features are characteristic of a system with only one degree of freedom.

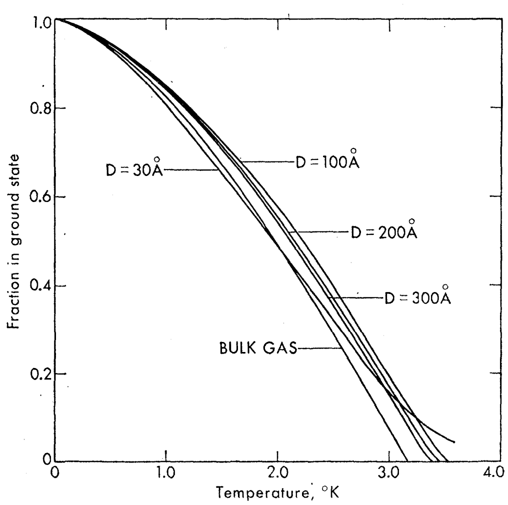

FIG. 1. Occupation of the ground state as a function of temperature for $L=200$ Å.

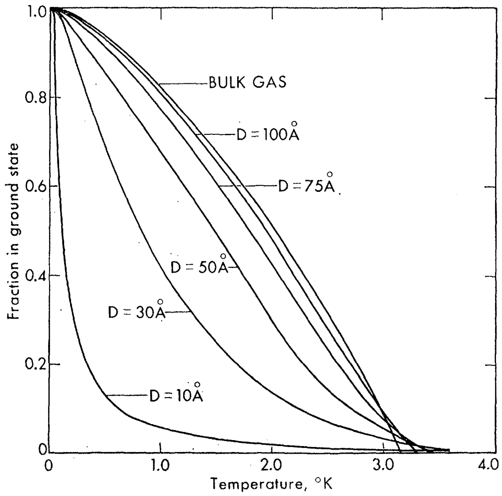

FIG. 2. Occupation of the ground state as a function of temperature for $L=1000$ Å.

Suppose that in the expression for $E_{n_1n_2n_3}$ we keep n_3 fixed and examine the spacing between the levels E_{11n_3} and E_{21n_3}. We find $E_{21n_3}-E_{11n_3}=\Delta=3\pi^2\hbar^2/2mD^2$. For $D=10$ Å, $\Delta/kT\cong1.8/T$, and for $D=100$ Å, $\Delta/kT\cong(1.8\times10^{-2})/T$.

Thus, for $D=10$ Å we see that the levels for which n_1 or n_2 are >1 are separated from the ground state by an energy roughly equal to kT for T in the range of 1 to 2°K. Since there are a large number of levels of the type E_{11n_3} which lie within kT of the ground state, we expect levels for which n_1 or n_2 are >1 to have little influence on the properties of the system. Thus, we expect the system to behave like a one dimensional gas of bosons confined to a line of length L. It will be shown below that an initial rapid depopulation of the ground state and a nonexponential tail are properties of such a one dimensional gas.

The allowed energy eigenvalues for a particle confined to a one dimensional line of length L are $E_n=\gamma n^2$, where $\gamma=\pi^2\hbar^2/2mL^2$ and $n=1,2,3,\cdots$. Then, as before,

$$N=\sum_{n=1}^{\infty}\left[\exp\left(\frac{\gamma n^2-\mu}{kT}\right)-1\right]^{-1}. \quad (4)$$

Eq. (4) may be written in the form

$$N=N_1+\sum_{n=2}^{\infty}\sum_{\lambda=0}^{\infty}\left(\frac{N_1}{N_1+1}\right)^{\lambda+1}\times\exp\left[-\frac{(\lambda+1)(n^2-1)}{kT}\gamma\right], \quad (5)$$

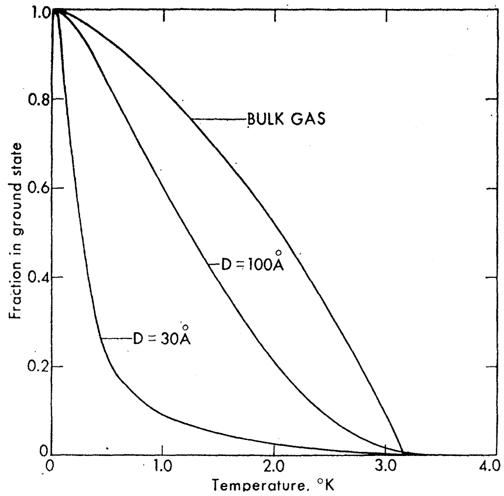

FIG. 3. Occupation of the ground state as a function of temperature for $L=5000$ Å.

where N_1 is the number of particles in the ground state. In the very high temperature limit, $N_1 \ll 1$ and $\gamma \ll kT$. Then only the $\lambda=0$ term need be retained on the right-hand side of Eq. (5) and the summation on n may be replaced by integration. We find

$$N_1/N=\alpha(T)=(2\gamma/\pi kT)^{1/2}.$$

If $L=100$ Å and $N=10^3$, the above approximations are valid for $T \gg 50$°K. Thus, at $T \rightarrow \infty$, $\alpha(T) \propto 1/T^{1/2}$ and we see that $\alpha(T)$ has a long nonexponential tail.

In the low-temperature limit $N_1 \gg 1$ if $N \gg 1$. Then in Eq. (5) we have $N_1/(N_1+1) \cong 1$, and we find

$$\alpha(T)=1-\frac{1}{N}\sum_{n=2}^{\infty}\left[\exp\left(\frac{\gamma(n^2-1)}{kT}\right)-1\right]^{-1}.$$

The temperature $T_{0.9}$ at which $\alpha(T)=0.9$ may be found from

$$\sum_{n=2}^{\infty}\left[\exp\left(\frac{\gamma(n^2-1)}{kT_{0.9}}\right)-1\right]^{-1}=0.1N. \tag{6}$$

We obtain an upper bound on $T_{0.9}$ by keeping only the first term on the left-hand side of Eq. (6). This yields

$$T_{0.9} \leq \frac{0.15\pi^2\hbar^2 nD^2}{mkL}.$$

For $D=10$ Å and $L=100$ Å we find $T_{0.9}=0.03$°K. The computer calculations of Fig. 2 yields $T_{0.9} \cong 0.05$°K for this case. For the bulk gas $T_{0.9} \cong 0.70$°K.

Thus, the rapid initial depopulation of the ground state and the long nonexponential tail are characteristic of the one-dimensional gas.

The results for $L=5000$ Å are given in Fig. 3. We see that for a given value of D the curve for $\alpha(T)$ is depressed considerably more than the corresponding curve for $L=1000$ Å. In particular the curve for $L=5000$ Å and $D=100$ Å shows large deviations from the bulk-gas curve even though $\Delta/kT \ll 1$ for T in the range of 1 to 2°K.

Even if $\Delta/kT \ll 1$ the system can be expected to behave like a one-dimensional system if the number of states of the type E_{11n_3} in a given energy range greatly exceeds the number of states of the type $E_{n_1n_21}$ in the same energy range. If we treat the eigenvalue spectrum as a continuum, the density of states of type E_{11n_3} is the same as the total density of states of a one-dimensional line of length L, while the density of states of type $E_{n_1n_21}$ is the same as that for a two-dimensional square of side D. Let $\rho_1(E)$ denote the total density of states of the one-dimensional line and let $\rho_2(E)$ denote the total density of states of the square. We have $\rho_1(E)=(L/2\pi\hbar)(m/2E)^{1/2}$ and $\rho_2(E)=mD^2/2\pi\hbar^2$. We expect large deviations from bulk gas behavior when

$$\frac{\rho_2(kT)}{\rho_1(kT)}=\frac{D^2}{\hbar L}(2mkT)^{1/2} \ll 1. \tag{7}$$

From Fig. 3 it may be seen that significant deviations from bulk-gas behavior occurs when $T=0.20$°K for $D=100$ Å. Inserting $T=0.20$°K and $L=5000$ Å into our criteria (7), we see that such differences are expected when $D \ll 140$ Å. Eq. (7) predicts deviations from bulk behavior for $D \ll 60$ Å if $L=1000$ Å and $T=0.20$°K. Hence, the deviations from bulk-gas behavior which occur when $\Delta/kT \ll 1$ result from the numerical predominance of levels of the type E_{11n_3}.

In Figure 1 the results for $L=200$ Å and various values of D from 30 to 300 Å are shown. The curve for $D=30$ Å and $L=200$ Å lies above the curve for $D=30$ Å and $L=1000$ Å, as expected from equation (7). The curve for $D=100$ Å and $L=200$ Å is found to lie above the bulk-gas curve by a considerable amount. The computations for $D=200$ Å and $D=300$ Å shown in Fig. 1 indicate that the results converge to the bulk-gas curve with increasing D.

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